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Compositeness effects, Pauli's principle and entanglement

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Abstract
We analyse some compositeness effects and their relation with entanglement. We show that the purity of a composite system increases, in the sense of the expectation values of the deviation operators, with large values of the entanglement between the components of the system. We also study the validity of Pauli's principle in composite systems. It is valid within the limits of application of the approach presented here. We also present an example of two identical fermions, one of them entangled with a distinguishable particle, where the exclusion principle cannot be applied. This result can be important in the description of open systems.

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1. Introduction

According to the usual textbook rule, when in a physical process the internal structure of a composite particle is not revealed; the particle should approximately behave as a boson or a fermion depending on the number of constituent fermions and bosons. When the internal structure of the particle is taken into account, compositeness effects can appear which manifest in deviations from the purely bosonic or fermionic behaviour. These deviations have been studied in several contexts. From a fundamental point of view, the interactions between composite bosons have been analysed highlighting the differences with the case of pure bosons [1, 2]. On the other hand, several systems where these deviations can take place have been studied from both the theoretical and experimental points of view. There are two principal lines: Bose–Einstein condensation (BEC) and semiconductors. The BEC occurs even when the atoms of a condensate gas are not pure bosons. Small deviations of the pure behaviour in a BEC have been presented in [3]. In the case of semiconductors, the research has focused on excitons (electron–hole bound pairs). Criteria for the bosonic behaviour of excitons have been derived in [4]. From an experimental point of view, the excitons can be studied by optical spectroscopy (photoluminescence, reflectivity, etc). In particular, there has been an extensive analysis of the optical spectra of quantum wells (two-dimensional structures containing electron gases [5]). We must also mention the issue where both lines, BEC and semiconductors, converge, the BEC of excitons. It was predicted by Keldysh and Zozlov [6], being quantum degeneracy experimentally observed in the system later [7].

More recently, the close relation between entanglement and the purity of a composite boson has been signalled [8]. That analysis was based on the properties of the creation and annihilation operators of the composite boson. In this paper, we show with a particular example that a similar result is obtained when the purity of the composite boson is measured in terms of the expectation values of the deviation operators.

In addition, and as the main aim of the paper, we study in a rigorous way the validity of Pauli's principle in 'composite fermions' of the type considered in this paper. To our knowledge, this issue has not previously been considered in the literature. We shall show that in the range where the theory considered here is applicable the exclusion principle remains exactly valid for composite particles. On the other hand, we shall present an example where if one of two identical fermions in the same state becomes entangled with a third distinguishable particle the exclusion principle does no longer act between them. This result is potentially important in open systems, where fermions can interact with a large number of particles. If these interactions are of the type that lead to entanglement the exclusion principle can become fragile. This result shows striking resemblance with decoherence theory [9].

We shall restrict our considerations to systems composed of two particles in multimode states. The starting point of the analysis will be the (anti)commutation relations of creation and annihilation operators. We shall present the set of (anti)commutation relations in a general way, including those of composite particles in different multimode states.

The plan of the paper is as follows. In section 2 we present the general set of (anti)commutation relations. Section 3 deals with an analytic example of the evaluation of the expectation values of the deviation operators. In sections 4 and 5 we study, respectively, the validity of the exclusion principle in composite systems and in the presence of entanglement with other particles. In section 6 we emphasize on the main results of the paper. Finally, in the appendix we present an alternative derivation of the results of section 5.

2. The (anti)commutation relations

The wavefunction of a system composed of two distinguishable particles in continuous multimode states is given by

$$\psi(x, y) = \int\int \mathrm{d}p\, \mathrm{d}q\, f(p, q)\Psi_{p,r}(x)\phi_{q,s}(y) \tag{1}$$

where x and y are the coordinates associated with the two particles (for simplicity, we only consider the one-dimensional problem). $\Psi_{p,r}$ and $\phi_{q,s}$ are the wavefunctions corresponding to the modes with momentum p and q in spin states r and s (the same spin state for all the modes of every particle). On the other hand $f(p, q)$ is the distribution of modes. We assume for simplicity that the distribution of modes is independent of the spin states. The integrations in equation (1) extend between $-\infty$ and ∞ (just as all the other integrals appearing in the paper).

In the second quantization formalism, the composite particle represented by equation (1) corresponds to the state generated by the creation operator

$$\hat{c}_f^+ = \int\int \mathrm{d}p\, \mathrm{d}q\, f(p, q)\hat{a}_{p,r}^+\hat{b}_{q,s}^+ \tag{2}$$

where $\hat{a}_{p,r}^+$ and $\hat{b}_{q,s}^+$ are the creation operators of modes $\Psi_{p,r}$ and $\phi_{q,s}$ (as signalled before we assume both particles to be distinguishable ones).

After simple calculations using the relations $\left[\hat{a}_{p,r}, \hat{a}^{+}_{P,s}\right] = \delta(p-P)\delta_{rs}$, $\left[\hat{a}^{+}_{p,r}, \hat{a}^{+}_{P,s}\right] = 0$ and $[\hat{a}_{p,r}, \hat{a}_{P,s}] = 0$ for bosons and $\left\{\hat{a}_{p,r}, \hat{a}^{+}_{P,s}\right\} = \delta(p-P)\delta_{rs}$, $\left\{\hat{a}^{+}_{p,r}, \hat{a}^{+}_{P,s}\right\} = 0$ and $\{\hat{a}_{p,r}, \hat{a}_{P,s}\} = 0$ for fermions, the (anti)commutation relations between the creation and annihilation operators of composed particles with different mode distributions f and g are as follows (we denote the spin states of particles a and b in mode f by r and s, and those in mode g by R and S, respectively),

$$[\hat{c}_f, \hat{c}_g]_{\mathrm{BB}} = [\hat{c}_f, \hat{c}_g]_{\mathrm{FF}} = \{\hat{c}_f, \hat{c}_g\}_{\mathrm{FB}} = 0 \tag{3}$$

$$\left[\hat{c}^{+}_f, \hat{c}^{+}_g\right]_{\mathrm{BB}} = \left[\hat{c}^{+}_f, \hat{c}^{+}_g\right]_{\mathrm{FF}} = \left\{\hat{c}^{+}_f, \hat{c}^{+}_g\right\}_{\mathrm{FB}} = 0 \tag{4}$$

$$\left[\hat{c}_f, \hat{c}^{+}_g\right]_{\mathrm{BB}} = \theta\hat{1} + \hat{\theta}_a + \hat{\theta}_b \tag{5}$$

$$\left[\hat{c}_f, \hat{c}^{+}_g\right]_{\mathrm{FF}} = \theta\hat{1} - \hat{\theta}_a - \hat{\theta}_b \tag{6}$$

and

$$\left\{\hat{c}_f, \hat{c}^{+}_g\right\}_{\mathrm{FB}} = \theta\hat{1} - \hat{\theta}_a + \hat{\theta}_b \tag{7}$$

with

$$\theta = \iint \mathrm{d}p\,\mathrm{d}q\; f^*(p,q)g(p,q)\delta_{rR}\delta_{sS} = \langle f|g\rangle\delta_{rR}\delta_{sS} \tag{8}$$

$$\hat{\theta}_a = \iiint \mathrm{d}p\,\mathrm{d}P\,\mathrm{d}q\; f^*(p,q)g(P,q)\hat{a}^{+}_{P,R}\hat{a}_{p,r}\delta_{sS} \tag{9}$$

and

$$\hat{\theta}_b = \iiint \mathrm{d}p\,\mathrm{d}q\,\mathrm{d}Q\; f^*(p,q)g(p,Q)\hat{b}^{+}_{Q,S}\hat{b}_{q,s}\delta_{rR}. \tag{10}$$

In the above expressions symbols [,] and { , } refer, respectively, to commutators and anticommutators. On the other hand, BB, FF and FB denote particles composed of two bosons, two fermions or a fermion and a boson. In the case of boson–fermion systems we adopt the convention that a refers to the fermion and b to the boson. Variable θ has been expressed in terms of $\langle f|g\rangle$, the scalar product of wavefunctions f and g in momentum space. This expression shows that when f is orthogonal to g, θ vanishes.

We obtain a set of relations that differs from the pure bosonic and fermionic algebras. Only when $\theta = 1$, $\hat{\theta}_a = 0$ and $\hat{\theta}_b = 0$ can we recover the usual (anti)commutation relations for bosons and fermions. This property resembles that of the quon algebra, which interpolates between the bosonic and fermionic ones [10] (it has been used in [3] to study compositeness effects).

We note that these commutation relations are different for boson–boson (BB) and fermion–fermion (FF) systems.

It is also simple to see that $\hat{\theta}_a$ and $\hat{\theta}_b$ are not, in general, nulls, even in the absence of common modes. Consequently, the (anti)commutator of the creation and annihilation operators with different distributions can be different from zero even in the case when they have no common modes. This result does not follow the intuition obtained in the study of systems in multimode states. For instance, in [11] it was shown that the existence of common modes is a necessary condition for the existence of interference effects in the arrangement considered in that paper. This property could, in principle, be experimentally tested in systems whose interactions depend on the (anti)commutator of the creation and annihilation operators.

3. Expectation values of deviation operators

One of the ways of characterizing the importance of the new effects in composite systems is via the expectation values of the deviation operators. We say that a system behaves as a pure boson (fermion) when their creation/annihilation operators obey the Bose (Fermi) commutation (anticommutation) relations. When a composite system does not obey these relations the departure from the pure behaviour can be quantitatively estimated by some measure of the difference between the relations of pure and composite systems. As remarked in the previous section, operators $\theta\hat{1}$ (different from $\hat{1}$), $\hat{\theta}_a$ and $\hat{\theta}_b$ (different from 0) are responsible for the departures from the usual algebras, and can be named deviation operators. The importance of the deviation operators depends on the state of the system. We must evaluate their expectation values on the state of the system. We use these expectation values as a measure of the degree of purity of the system (how close the system is to a pure one).

In order to illustrate the general method, we shall evaluate in this section these expectation values for two identical 'composite bosons'. The distributions and spin states must also be equal ($f = g$, $r = R$ and $s = S$), and are no longer necessary to include explicitly the spin indexes in the expressions. The state of the system is $|2_f\rangle = N(c_f^+)^2|0\rangle$, with N the normalization factor, given by

$$2N^2_{\substack{\mathrm{BB}\\\mathrm{FF}}} = \frac{1}{1 \pm \wedge} \tag{11}$$

where

$$\wedge = \iiiint \mathrm{d}p\,\mathrm{d}P\,\mathrm{d}q\,\mathrm{d}Q\; f^*(P,Q) f(P,q) f^*(p,q) f(p,Q). \tag{12}$$

In this expression and from now on, in all the double expressions the upper sign corresponds to the BB case and the lower one to the FF one.

To obtain the above equations we have used the normalization of the distributions, $\iint \mathrm{d}p\,\mathrm{d}q |f(p,q)|^2 = 1$. This normalization emerges directly from the usual normalization of the wavefunction, $\iint \mathrm{d}x\,\mathrm{d}y |\psi(x,y)|^2 = 1$, and the orthogonality relations between Ψ and ϕ.

In this case, since both particles are equal we have $f = g$ and using the normalization relation we obtain $\theta = 1$. The expectation values are

$$\langle 2_f|\hat{\theta}_a|2_f\rangle_{\substack{\mathrm{BB}\\\mathrm{FF}}} = \langle 2_f|\hat{\theta}_b|2_f\rangle_{\substack{\mathrm{BB}\\\mathrm{FF}}} = N^2_{\substack{\mathrm{BB}\\\mathrm{FF}}} \times \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \mathrm{d}p\,\mathrm{d}P\,\mathrm{d}p_*\,\mathrm{d}q\,\mathrm{d}Q\,\mathrm{d}q_*\; F(p,\ldots,q_*) G(p,\ldots,q_*) \tag{13}$$

where

$$F(p,\ldots,q_*) = f^*(p,q) f^*(P,Q) f^*(p_*,q_*) \tag{14}$$

and

$$G(p,\ldots,q_*) = 2f(p_*,Q) f(P,q_*) f(p,q) + 2f(p_*,q) f(p,q_*) f(P,Q) \pm 2f(p_*,q) f(p,Q) f(P,q_*) \pm 2f(p_*,Q) f(P,q) f(p,q_*). \tag{15}$$

To illustrate the behaviour of the above expressions we choose a tractable example of distribution,

$$f(p,q) = \sqrt{\frac{2}{\pi}} (\alpha\beta - \gamma^2)^{1/4} \exp(-\alpha p^2 - \beta q^2 - 2\gamma pq) \tag{16}$$

with $\alpha > 0$ and $\beta > 0$, both real for simplicity. Taking $\alpha\beta \geqslant \gamma^2$ this distribution is a Gaussian one. We note that the distribution is normalized. In the absence of the third term in the exponential, α^2 and β^2 are the widths along directions p and q, measuring the spread of the distribution. When $\gamma = 0$ the distribution can be separated into two independent distributions of p and q. If $\gamma \neq 0$ the distribution is no longer separable. As the distribution f is the wavefunction in momentum space, $\gamma \neq 0$ can be associated with the presence of entanglement in the system. This can be directly verified by transforming to the position space where, for $\alpha = \beta$, we have $\psi(x, y) \sim \exp(-(\alpha x^2 + \alpha y^2 + 2\gamma xy)/4(\alpha^2 - \gamma^2)\hbar^2)$. We see again that for $\gamma \neq 0$ the wavefunction cannot be separated. A measure of the degree of entanglement is $\gamma^2/\alpha\beta$. For $\gamma = 0$ the entanglement vanishes. On the other hand, for $\gamma^2 \to \alpha\beta$ the expression tends towards the maximum value, 1, which for $\alpha = \beta$ corresponds to $\gamma \to \pm\alpha$, which gives $\alpha(p^2 + q^2) \pm 2\gamma pq \to \alpha(p \pm q)^2$. The distribution has only appreciable values for $p \to \pm q$ ($|p| \approx |q|$).

For this particular distribution, the normalization and expectation values become

$$2N^2_{\substack{\mathrm{BB}\\ \mathrm{FF}}} = \frac{1}{1 \pm \epsilon} \tag{17}$$

and

$$\langle 2_f|\hat{\theta}_a|2_f\rangle_{\substack{\mathrm{BB}\\ \mathrm{FF}}} = \langle 2_f|\hat{\theta}_b|2_f\rangle_{\substack{\mathrm{BB}\\ \mathrm{FF}}} = \frac{2\epsilon}{1 \pm \epsilon} \pm \frac{8\epsilon^3(\alpha\beta)^{3/2}}{\sqrt{2\mu\eta}(2\alpha\beta - \gamma^2)^{1/2}(1 \pm \epsilon)} \tag{18}$$

with

$$\epsilon = \left(1 - \frac{\gamma^2}{\alpha\beta}\right)^{1/2} \tag{19}$$

$$\mu = 2\alpha\beta - \gamma^2 - \frac{\gamma^4}{4(2\alpha\beta - \gamma^2)} \tag{20}$$

and

$$\eta = \mu - \frac{1}{4\mu}\left(\gamma^2 + \frac{\gamma^4}{2(2\alpha\beta - \gamma^2)}\right)^2. \tag{21}$$

The new parameter ϵ is the square root of $1 - \frac{\gamma^2}{\alpha\beta}$. As $\frac{\gamma^2}{\alpha\beta}$ is a measure of the entanglement of the system ϵ will be a measure of its complementary variable, i.e., of the degree of separation of the state. When $\gamma = 0$, $\epsilon = 1$ and the state can be completely separated. On the other hand, when $\gamma^2 = \alpha\beta$, $\epsilon = 0$ reaching its minimum separation.

In order to carry out these integrals with the expression $\int \mathrm{d}x \exp(ax^2 + bx) = (-\pi/a)^{1/2} \exp(-b^2/4a)$, a must obey the relation $a \leqslant 0$. In the case of equation (17) this relation gives

$$2\alpha - \frac{\gamma^2}{\beta} \geqslant 0, \qquad 2\alpha - \frac{\gamma^2}{\beta} - \frac{\gamma^4}{\beta(2\alpha\beta - \gamma^2)} \geqslant 0. \tag{22}$$

It is simple to see that these relations automatically hold when $\alpha\beta \geqslant \gamma^2$. In the case of equation (18) the relations are

$$2\alpha - \frac{\gamma^2}{\beta} \geqslant 0, \qquad \frac{\mu}{\beta} \geqslant 0, \qquad \frac{\mu}{\beta} - \frac{\left(\gamma^2 + \frac{\gamma^4}{2(2\alpha\beta - \gamma^2)}\right)^2}{4\beta\mu} \geqslant 0. \tag{23}$$

As in the previous case it is simple to show after some manipulations that all these relations are valid when $\alpha\beta \geqslant \gamma^2$.

When $\gamma^2 \to \alpha\beta$ (the maximum entanglement of the system) the expectation values of the deviation operators tend to zero for both BB and FF systems. The composite systems behave, in the sense of expectation values, as pure bosonic systems. On the other hand, when $\gamma^2 \neq \alpha\beta$ the expectation values are, in general, different from zero.

This result agrees with those of [8], where it was demonstrated that bosonic character emerges when the constituent particles become strongly entangled. It must be remarked that in that reference the results were derived studying the properties of the creation and annihilation operators of the composite particles. In this paper, we have based our analysis on the expectation values of the deviation operators, providing an independent confirmation of the connection between almost pure bosonic behaviour and strong entanglement.

It must also be remarked that the expectation values are different for FF and BB systems because of the different signs in the expressions of type $\pm$. This property is specially striking for completely decorrelated particles ($\gamma = 0$). This limit will be considered in the next section.

4. Pauli's principle in composite systems

We devote sections 4 and 5 to the analysis of Pauli's principle in composite systems. First, in this section, we study the behaviour of identical 'composite fermions'.

An important consequence concerning Pauli's principle follows directly from the relations derived in section 2. From the anticommutation relations in equation (4) we have $\left(\hat{c}_f^+\right)^2 = 0$, which is the mathematical expression of the exclusion principle. Effectively, if we try to create a state with two identical 'composite fermions' we must apply the operator $\left(\hat{c}_f^+\right)^2$ on the vacuum $|0\rangle$ with the result $\left(\hat{c}_f^+\right)^2|0\rangle = 0$. It is impossible to prepare such a state. In addition to the case of systems composed of two particles, it is simple to verify by direct calculation that this property is valid for any type of particle composed of an odd number of (distinguishable) fermions. We conclude that it is impossible to prepare two or more 'composite fermions' of the type considered here in the same state. Where the results presented here are valid they provide a justification for the use of the principle in non-pure conditions, although the complete set of anticommutation relations differs from the pure one (we do not have the pure fermion algebra).

Of course, the validity of this result is limited by the scope of the framework considered here. Essentially, this limit is given by the validity of the description of the two-particle system as a single entity in the second quantization formalism. Although mathematically this procedure can always be carried out it can be misleading from a physical point of view.

In order to clarify this point let us consider an example. The second quantization formalism is relevant for the problem when the creation/annihilation operators represent correctly the physical processes involved. Let us consider, for instance, the annihilation operator and a situation where a composite particle interacts with an absorbing medium, i.e., with one that can capture and absorb particles (the composite one and/or any of its two components a and b). If, for instance, the medium can absorb particle a without absorbing b (and, consequently, without absorbing the composite system) the annihilation operator of particle a is relevant for the problem, but not that of the composite system. In this case, the description of the composite system in the second quantization formalism as a single entity represented by only an annihilation operator is meaningless. For bound systems with the two particles a and b close one expects, in general, the operator of the composite system to be relevant. However, for entangled separated systems we have no *a priori* general criterion for this problem. The analysis of the relevance of the annihilation (or creation) operator must be carried out for every system.

5. Pauli's principle and entanglement

A detailed analysis of the example presented in section 3 in the limit of no correlation in the mode distributions shows an interesting connection between entanglement and Pauli's principle. We devote this section to study this connection. We start our discussion by considering that example.

5.1. An example

We see that for FF systems expression (17) becomes unbounded (and then physically forbidden) when the distribution can be factored, i.e., when $\gamma = 0$ and the distributions of the two particles are uncorrelated. Moreover, equation (18) shows that when $\gamma = 0$, the expectation values of the deviation operators of the composed FF systems become of the undetermined form $0/0$. These results reflect the impossibility of preparing the system in such a state. No counterpart of such critical behaviour is found for BB systems, for which a finite value is reached.

These results are manifestations of Pauli's principle, which acts in the absence of entanglement ($\gamma = 0$). When the particles are not entangled, we have two free indistinguishable fermions of every class (a and b). Pauli's principle acts between them. However, when there is entanglement the exclusion principle does no longer act. We find again a close relationship between the behaviour of composite systems and entanglement.

This property is not only valid for the distribution considered here. For any pair of composite particles of the FF type in a non-entangled state the distribution $f(p, q)$ is separable in the form $f_a(p) f_b(q)$. Then using the normalization condition of the distributions we have $\wedge = 1$. The normalization constant is unbounded reflecting the impossibility of preparing the state (as a consequence of Pauli's principle). However, if $f(p, q)$ is non-separable, in general, $\wedge \neq 1$ and the normalization constant is finite.

5.2. Two fermions, one in an entangled state

These results are only an example of a more general and important property. Let us imagine two identical non-composite fermions. Initially they are very separated (then the overlapping between their wavefunctions is negligible and it is not necessary to antisymmetrize the complete wavefunction [12]). One of them can interact with a third particle of a different type. The interaction will result in many cases in an entangled state of the fermion and the third particle (the interaction Hamiltonian can in many cases couple both particles). Let us consider the case when this pair of entangled particles must be described as a single entity in the second quantization formalism (see the discussion at the end of the previous section). Later, the two fermions come together. A particular realization of this scheme (of a Gedanken type) is as follows. The distinguishable particle and one of the indistinguishable ones are enclosed in a box where they interact and become entangled. On the other hand, the other indistinguishable particle is placed in another box. Both boxes are separated by a common movable wall, which prevents any overlapping between the identical particles (as remarked before, without overlapping it is not necessary to take into account the indistinguishable character of the particles [12]). Later, we remove the wall and the overlapping of the indistinguishable particles becomes appreciable. If necessary, before removing the wall we displace the distinguishable particle in order for the interaction with the indistinguishable particle (that previously in the other box) to be negligible.

We assume that the state of the free fermion and the local state (obtained by local measurements on the fermion) of the entangled fermion are the same. Note that an observer can be unaware of the interaction with the third particle, consequently identifying the local state of the entangled fermion with the state of the fermion. We study if the exclusion principle can be applied in this case.

The three-particle state is $|3\rangle = N_3 \hat{c}_f^+ \hat{a}_g^+ |0\rangle$ with $\hat{c}_f^+$ and $\hat{a}_g^+$ the creation operators of the composite particle and the free fermion, respectively. The normalization constant is given by

$$N_3^{-2} = \iiint \mathrm{d}p\, \mathrm{d}P\, \mathrm{d}q (g^*(p) f^*(P,q) g(P) f(p,q) - g^*(p) f^*(P,q) g(p) f(P,q))$$

$$= -1 + \iiint \mathrm{d}p\, \mathrm{d}P\, \mathrm{d}q\, g^*(p) f^*(P,q) g(P) f(p,q). \qquad (24)$$

When there is no interaction between the fermion and the third particle or the interaction does not induce entanglement, $f(p,q)$ can be factored into the form $g(p)F(q)$ (with the two identical fermions having the same mode distribution $g(p)$). N_3 becomes unbounded and the state of the system cannot be normalized. Without normalization the state cannot be interpreted along the usual statistical formulation of quantum theory. Consequently, it is not a physical state that can be associated with the system in the framework of standard quantum theory. In other words, it is not a state that can be prepared following the usual schemes. Pauli's principle acts between the two fermions, avoiding the possibility of preparing the three particles in that state. On the other hand, when the particles become entangled the mode distribution cannot be factored and, in general, the rhs of the above equation is not zero[1]. The state admits the usual statistical interpretation and the system can be prepared in that state. Note that this result is independent of the spin states of the three particles (of course, the two identical fermions are in the same one).

One can think that the use of boundless integrations could be responsible for the appearance of infinite values in the normalization of these states. In particular, the use of Dirac's delta in the calculations involving (anti)commutators can be suspicious in this respect. To discard this possibility, we have repeated the calculations in a finite volume version of the problem. The integration is replaced by sums over discrete indexes related to the quantized momenta of the particles. Now the creation operator of the composite particle is $\hat{a}_f^+ = \sum_{n,m} f_{nm} \hat{a}_n^+ \hat{b}_m^+$ with f_{nm} the coefficient of the composite mode nm, with normalization $\sum_{n,m} f_{nm}^* f_{nm} = 1$. The expressions for the (anti)commutators are $\left[\hat{b}_n, \hat{b}_m^+\right] = \delta_{nm}$ and $\left\{\hat{a}_n, \hat{a}_m^+\right\} = \delta_{nm}$. After a simple calculation we have $N_3^{-1} = -1 + \sum_{n,m,r} f_{nm}^* f_{rm} g_r^* g_n$. The same conclusions of the continuous case are valid for the discrete one. It is simple to verify by direct calculation that all the other results derived in this paper remain valid, with obvious modifications, for discrete systems.

The result of this section has been derived in the second quantization formalism. In the appendix we show that the same conclusion is obtained in the more familiar first quantization formalism.

In conclusion, the example considered here shows that in some scenarios the existence of entanglement between a fermion and a distinguishable particle can preclude the application of the exclusion principle between two identical fermions. Next we shall study the scope of this result.

5.3. Scopes of the method and the result

It is important to correctly understand the scopes of the above method and its result in order to know when the exclusion principle can or cannot be applied in the presence of entanglement. The first limitation comes from the assumption of the system of entangled particles to be described as a single entity in the second quantization formalism. As discussed in the previous

[1] This statement can be explicitly demonstrated in the case of Gaussian distributions. For the distributions (16) and $g(p) = (2\alpha/\pi)^{1/4} \exp(-\alpha p^2)$ the integral gives $\sqrt{2}(\alpha\beta - \gamma^2)^{1/2}(2\alpha\beta - \gamma^2)^{-1/2}$, expression that only equals 1 for $\gamma = 0$.

section, from a physical point of view, this assumption is not plausible in some situations. We must analyse in every particular scenario when it is and when it is not.

The framework considered above excludes the case when the third particle with which one of the fermions interacts is an indistinguishable one. The analysis of this problem is much more involved than that presented here. When the third particle is indistinguishable of the pair of fermions there are two types of entanglement, the dynamical entanglement associated with interactions between the particles and the statistical entanglement originated with the indistinguishable nature of the particles. For instance, in [13] the existence of entanglement in an ideal (without interaction) gas of identical fermions has been shown. Then both types of entanglement must be considered separately, becoming necessarily a different type of analysis.

An example within the range of applicability of our method, but with an opposite result, is that of several identical fermions interacting with other distinguishable particles in bound states, for instance, electrons in atoms. The three-particle case corresponds to the helium atom (in the approximation that the nucleus can be treated as a single particle, although it is a composite one). Now there is simultaneous interaction between the three particles. The three-particle system can no longer be decomposed into a free fermion and an entangled two-particle system. The momentum distribution cannot be separated in the form $f(p,q)g(P)$. The three-particle state now has the form

$$|\tilde{3}\rangle = N_{\tilde{3}} \iiint \mathrm{d}p\,\mathrm{d}q\,\mathrm{d}P\; f(p,q,P)\hat{a}^{+}_{p,r}\hat{a}^{+}_{q,s}\hat{b}^{+}_{P,S}|0\rangle. \tag{25}$$

We explicitly include the spin states for the sake of clarity. The normalization constant is

$$N_{\tilde{3}}^{-2} = \iiint \mathrm{d}p\,\mathrm{d}q\,\mathrm{d}P\; f^{*}(p,q,P)(f(p,q,P) - f(q,p,P)\delta_{rs}). \tag{26}$$

For symmetric interactions between the three particles (as in the case of atoms where all the interactions depend on the positions of the particles in the form $|\mathbf{r}_i - \mathbf{r}_j|$) we have $f(p,q,P) = f(q,p,P)$, as can be easily seen by transforming the wavefunction to the momentum representation. Then for fermions in the same state, $r = s$, we have $N_{\tilde{3}}^{-2} = 0$. The exclusion principle can be applied to the system although there is entanglement. On the other hand, when $r \neq s$ we have a finite normalization constant and the state is physically admissible. We recover the usual results for these types of systems. We see that our method to decide if the exclusion principle must be applied to a particular state, based on the normalization constant of the state, accounts for three-particle states such as atoms.

To sum up, the analysis presented in this subsection establishes the scopes of the method and the result presented in the previous subsection. Our method can only be applied to pairs of fermions that interact with distinguishable particles. This excludes some important scenarios as the ideal gas. On the other hand, if the interaction is with a distinguishable particle our result is valid when the third particle is only entangled with one of the two fermions and, in addition, as discussed in the previous section, the entangled fermion-distinguishable particle system can be described as a single entity in the second quantization formalism.

6. Discussion

We have derived the general (anti)commutation relations of creation and annihilation operators of composite particles. We have also studied the role of Pauli's principle in composite systems. It is exactly valid for 'composite fermions' (in spite of the fact that the complete set of anticommutation relations differs from that of pure fermions) when the description of the two interacting particles as a single entity in the second quantization formalism is physically meaningful. As discussed in section 4 a good framework to discuss this physical relevance is that of the creation/annihilation operators. One must decide when the operators associated with the composite system provide a good description of the system and when the correct description is given by the operators of the component particles. For pairs of particles in bound states it seems intuitive that the composite system must be described as a composite particle. However, for other systems as entangled two-particle systems not in bound states the answer is not clear and must be elucidated for every particular system.

We have also presented an example of two identical fermions, one of them entangled with a distinguishable particle, where Pauli's principle cannot be applied. In our example the temporal sequence of the interactions is fundamental. The interaction with the distinguishable particle must be well separated in time from the overlapping of the two identical fermions (see also the discussion of the initial conditions in the appendix). This example shows that in these types of situations we must decide if the exclusion principle can or cannot be applied. The mathematical technique used in this paper is based on the normalization constant of the state. We have seen that this procedure also gives a correct description of atoms with two electrons or similar systems where Pauli's principle must be applied. As discussed in section 5.3 the problem of two fermions interacting with an indistinguishable particle has some peculiarities due to the simultaneous presence of dynamical and statistical entanglement. These particular characteristics must be considered in detail before analysing these types of problems (in particular the relation between entanglement and Pauli's principle) along the line of this paper.

The inhibition of the exclusion principle in the example discussed in this paper can be surprising. However, it only reflects a fundamental property of entangled systems, where the characteristics of the components of the complete system can be modified. In general, we cannot characterize them in the same way as when they are isolated (in non-entangled states). In some circumstances a particle in an entangled state can become distinguishable from an identical particle, in spite of the fact that in the absence of entanglement the two particles would be indistinguishable. In other words, in some circumstances (the helium atom is an example of a system where this property is not valid) a component of an entangled system loses partially its identity within the larger system. A well-known analogy useful to illustrate this point is the analysis of interference patterns of two-particle systems in entangled states, where the existence of entanglement prevents the formation of one-particle interferences (a fundamental characteristic of its behaviour when several alternative paths are available to the particle), which would be present in the absence of entanglement [14].

The relation between entanglement and Pauli's principle could be specially relevant for open systems. In open systems, where it is impossible to completely shield the two identical fermions, almost certainly any of the fermions will interact with particles of the environment. We identify, as usual in decoherence theory [9], all the particles external to the two fermions with the environment. If the environment contains particles that become entangled with any of the fermions the action of the exclusion principle can be suppressed (of course, in addition to this entanglement, the rest of conditions under which we have derived this result must hold). Pauli's principle is potentially fragile in open systems. The resemblance with decoherence theory is evident. In decoherence theory the entanglement with the particles of the environment locally suppresses the interference terms. Similarly, the exclusion principle can be fragile in the presence of entanglement with particles of the environment.

Finally, we remark that in this paper we have restricted our discussions to one-dimensional systems. However, it is simple to verify that all the results derived in the previous sections remain valid in three-dimensional systems. In this case, the creation operator of the composite particle is $\hat{c}_f^+ = \int d^3\mathbf{p} \int d^3\mathbf{q}\, f(\mathbf{p},\mathbf{q})\hat{a}^+_{\mathbf{p},r}\hat{b}^+_{\mathbf{q},s}$. Using the (anti)commutation relations $\left[\hat{a}_{\mathbf{p},r}\hat{a}^+_{\mathbf{q},s}\right] = \delta^3(\mathbf{p}-\mathbf{q})\delta_{rs},\ldots$ we can see by simple calculation that the (anti)commutation relations derived in section 2 stand (with obvious modifications). Similarly, using instead of equation (16) the distribution $f(\mathbf{p},\mathbf{q}) \sim \exp(-\alpha\mathbf{p}^2 - \beta\mathbf{q}^2 - 2\gamma\mathbf{p}\cdot\mathbf{q})$ we obtain the same type of relation between expectation values and entanglement described in section 3 Neither behaviour of the normalization constant used in section 5 is modified by the transition to three dimensions. In relation to the dimensionality of the system an interesting situation is the two-dimensional case. When the dynamics of some composites (as, for instance, a charged particle bound to a tube of magnetic flux) can be confined to two dimensions the system can exhibit fractional statistics [15]. Even in some cases it is possible to transmute continually the statistics of the system from the fermionic one to the bosonic one (see, for instance, [16] where the transmutation is obtained by varying the degree of entanglement between the two subsystems). It would be an interesting problem to study if the presence of composite particles modifies the description of these types of two-dimensional systems.

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Appendix

We derive now the results of section 5 in the framework of first quantization, where the indistinguishable character of the particles is explicitly taken into account. For any system of three particles with two of them indistinguishable fermions the total wavefunction of the complete system must be antisymmetrized with respect to the variables of the identical particles,

$$\Psi(\mathbf{x},\mathbf{y},\mathbf{z}) \rightarrow \Psi(\mathbf{x},\mathbf{y},\mathbf{z}) - \Psi(\mathbf{y},\mathbf{x},\mathbf{z}) \tag{A.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the coordinates of the identical particles and $\mathbf{z}$ those of the distinguishable one. For simplicity, we do not explicitly include the normalization coefficients or the spin variables.

We first consider the case of section 5.2. After the interaction of the distinguishable particle with one of the identical ones they become entangled in the form $\psi(\mathbf{x},\mathbf{z})$; solution of the equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m_I}\nabla_x^2 - \frac{\hbar^2}{2m_D}\nabla_z^2 + V(\mathbf{x},\mathbf{z})\right)\psi \tag{A.2}$$

with m_I and m_D the masses of the two types of particles and V the potential ruling the interaction between them.

On the other hand, the other identical particle obeys the free-particle equation

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m_I}\nabla_y^2\phi. \tag{A.3}$$

Later, when the overlapping of the wavefunctions of the identical particles is not negligible (for simplicity we assume other interactions between them to be negligible) equation (A.1) becomes

$$\psi(\mathbf{x},\mathbf{z})\phi(\mathbf{y}) - \psi(\mathbf{y},\mathbf{z})\phi(\mathbf{x}). \tag{A.4}$$

In general, this expression is different from zero, and the probability of finding the particles in that state is not null. However, if V does not entangle the particles the state becomes ($\psi(\mathbf{x},\mathbf{z}) = \phi_*(\mathbf{x})\phi_D(\mathbf{z})$, where subscripts $*$ and D denote, respectively, that they are solutions of the same free equation but with different initial conditions and mass)

$$\phi_*(\mathbf{x})\phi_D(\mathbf{z})\phi(\mathbf{y}) - \phi_*(\mathbf{y})\phi_D(\mathbf{z})\phi(\mathbf{x}). \tag{A.5}$$

When $\phi_* = \phi$ (the identical fermions are in the same state) the expression becomes null: the probability of finding the fermions at the same position is zero.

When interaction V entangles the particles in the temporal order specified here, the exclusion principle does not, in general, act between the indistinguishable particles.

The situation is completely different in the helium atom (as an example of section 5.3). The ruling equation is now

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m_I}\left(\nabla_x^2 + \nabla_y^2\right) - \frac{\hbar^2}{2m_D}\nabla_z^2 + U(\mathbf{x},\mathbf{y},\mathbf{z})\right)\Psi. \tag{A.6}$$

Taking into account that all the terms are invariant under the interchange $\mathbf{x} \leftrightarrow \mathbf{y}$ (the Coulomb potential depends on $|\mathbf{x}-\mathbf{y}|$, just as the spin–spin interaction when it is taken into account), the Hamiltonian is invariant. Thus, if the initial conditions are also invariant under the interchange, $\Psi_0(\mathbf{x},\mathbf{y},\mathbf{z}) = \Psi_0(\mathbf{y},\mathbf{x},\mathbf{z})$ (note that in particular the two spins must be equal), we have $\Psi(\mathbf{x},\mathbf{y},\mathbf{z}) = \Psi(\mathbf{y},\mathbf{x},\mathbf{z})$ and the configuration is excluded by the exclusion principle.

The key role of the initial conditions in the above reasoning must be remarked. In the case of section 5.2 there is a delay between the entangling interaction and the overlapping of the identical fermions, and the initial wavefunction is not invariant under the exchange $\mathbf{x} \leftrightarrow \mathbf{y}$.

We finally note that we have obtained the same result in the first quantization framework by antisymmetrizing the wavefunction and in the second quantization one without antisymmetrizing the relevant states. This is so because in the second case the effects of antisymmetrization are generated by the anticommutation relations (see [17] for similar considerations in a different context).

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